Averaging for retarded functional differential equations

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Abstract

We consider retarded functional differential equations in the setting of Kurzweil-Henstock integrable functions and we state an averaging result for these equations. Our result generalizes previous ones.

1 Introduction

In [6] and [7], the authors stated very nice averaging results for retarded functional differential equations employing the tools of non-standard analysis. Their results encompass, for instance, the results by J. Hale and S. V. Lunel in [3].

In the present paper, we establish an averaging result for retarded functional differential equations, we write RFDE for short, by means of classical analysis. The conditions we assume on the righthand sides of the RFDEs are more general than those considered in [3], [6] or [7]. Indeed, we consider that the righthand sides of the equations are Kurzweil-Henstock integrable functions.

In the frame of the Kurzweil-Henstock integral, functions having not only many discontinuities but also being highly oscillating can be treated properly. It is known, for instance, that the Kurzweil-Henstock integral encompasses the integrals of Newton, Riemann and Lebesgue. In fact, the Kurzweil-Henstock integral coincides with the Perron and restricted Denjoy integrals and hence it can integrate functions as the well-known example $f(t) = \frac{d}{dt}F(t)$, where $F(t) = t^2 \sin t^{-2}$ on $[0, 1]$ when defined. Furthermore, the Kurzweil-Henstock integral is invariant by Cauchy and Harnack extensions and it has good convergence properties. See, for instance, [2], [5], [8], [9], [10] and the references therein.

Let $t_0 \in \mathbb{R}$, $r > 0$ and $\sigma > 0$. Given $t \in [t_0, t_0+\sigma]$ and a function $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, let $y_t : [-r, 0] \to \mathbb{R}^n$ be defined as usual by

$$y_t(\theta) = y(t + \theta), \qquad \theta \in [-r, 0].$$

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We consider the following initial value problem for an RFDE

$$\begin{cases} \dot{y} = f\left(y_t, \dfrac{t}{\varepsilon}\right) \\ y_0 = \phi, \end{cases}$$

where $\phi : [-r, 0] \to \mathbb{R}^n$ is a regulated function and $\varepsilon > 0$ is a small parameter. We assume that for every continuous function $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, the mapping $t \mapsto f(y_t, t)$ is Kurzweil-Henstock integrable, where $t \in [t_0, t_0 + \sigma]$ and the following condition holds:

(A) There is a constant $C > 0$ such that for $x, y \in PC_1$ and $u_1, u_2 \in [0, +\infty)$,

$$\left\| \int_{u_1}^{u_2} [f(y_s, s) - f(x_s, s)]\, ds \right\| \leq C \int_{u_1}^{u_2} \|y_s - x_s\|\, ds,$$

where PC_1 is a subset of the space of regulated functions from $[-r, 0]$ to $\mathbb{R}^n$, which we will specify later.

In this setting, we state an averaging result for the above RFDE (see Theorem 3.1 in the sequel).

Notice that we do not require that the function $f(\phi, t)$ is continuous. On the other hand, in [6] and [7], the authors assume that $f(\phi, t)$ is continuous and Lipschitzian in ϕ. In [3], the authors assume that $f(\phi, t)$ is almost periodic in t, uniformly with respect to ϕ in compact subsets of $\mathcal{C}([-r, 0], \mathbb{R}^n)$ and admits continuous Fréchet derivative in ϕ. (Hereafter $\mathcal{C}([-r, 0], \mathbb{R}^n)$ stands for the Banach space of continuous functions from $[a, b]$ to $\mathbb{R}^n$ equipped with the usual supremum norm, $\| \ \|_\infty$.

In the next section, we give some basic notation and definitions of the theory of Kurzweil-Henstock integration.

2 Basic facts of the Kurzweil-Henstock integral

Let $[a, b]$ be a compact interval of $\mathbb{R}$.

A *partial division* of $[a, b]$ is any finite set of closed non-overlapping intervals $[t_{i-1}, t_i] \subset [a, b]$. In this case we write $d = (t_i) \in PD_{[a,b]}$. If moreover $\cup_i [t_{i-1}, t_i] = [a, b]$, then d is a *division* of $[a, b]$ and we write $d = (t_i) \in D_{[a,b]}$. When $d = (t_i) \in PD_{[a,b]}$ and $\xi_i \in [t_{i-1}, t_i]$, for every i, then $d = (\xi_i, t_i)$ is called a *tagged partial division* of $[a, b]$, ξ_i is called the tag of the $[t_{i-1}, t_i]$, for each i. If in addition $d = (t_i) \in D_{[a,b]}$, then $d = (\xi_i, t_i)$ is a *tagged division* of $[a, b]$. We denote by $TPD_{[a,b]}$ the set of all tagged partial divisions of $[a, b]$ and by $TD_{[a,b]}$ the set of all tagged divisions of $[a, b]$.

A *gauge* of $[a, b]$ is any function $\delta : [a, b] \to]0, \infty[$. Given a gauge δ of $[a, b]$, $d = (\xi_i, t_i) \in TPD_{[a,b]}$ is called δ-*fine* whenever $[t_{i-1}, t_i] \subset \{t \in [a, b]\, ; |t - \xi_i| < \delta(\xi_i)\}$, for every i.

Definition 2.1. A function $f : [a, b] \to \mathbb{R}$ is *Kurzweil integrable* or *generalized Riemann integrable* (we write $f \in K([a, b], \mathbb{R})$) and $I = (K)\int_a^b f(t)\, dt \in \mathbb{R}$ is its integral, if given

$\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\left| (K) \int_a^b f(t)\,dt - \sum_i f(\xi_i)(t_i - t_{i-1}) \right| < \varepsilon.$$

In particular, when only constant gauges are considered in Definition 2.1, we obtain the well-known Riemann integral of a real valued function.

Now we define the integral for functions taking values in $\mathbb{R}^n$.

Definition 2.2. A function $f : [a,b] \to \mathbb{R}^n$ is *Kurzweil integrable* or *generalized Riemann integrable* (we write $f \in K([a,b], \mathbb{R}^n)$) and $I = (K) \int_a^b f(t)\,dt \in \mathbb{R}^n$ is its integral, if given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\left\| (K) \int_a^b f(t)\,dt - \sum_i f(\xi_i)(t_i - t_{i-1}) \right\| < \varepsilon,$$

where $\| \; \|$ denotes any norm in $\mathbb{R}^n$.

Clearly, any function $f : [a,b] \to \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$, is *Kurzweil integrable*, if and only if, every component f_i, $i = 1, \ldots, n$, of f is *Kurzweil integrable* in the sense of Definition 2.1.

We denote by $\tilde{f} : [a,b] \to \mathbb{R}$ the indefinite integral of a function $f \in K([a,b], \mathbb{R}^n)$, that is,

$$\tilde{f}(t) = (K) \int_a^t f(s)\,ds, \quad t \in [a,b].$$

If $f \in K([a,b], \mathbb{R}^n)$, then $\tilde{f} \in \mathcal{C}([a,b], \mathbb{R}^n)$ (see, for instance, [1], Theorem 2.2).

Let us present the definition of the Henstock integral of a real valued function.

Definition 2.3. A function $f : [a,b] \to \mathbb{R}$ is *variationally Henstock integrable* or simply *Henstock integrable* (we write $f \in H([a,b])$), if there is a function $F : [a,b] \to \mathbb{R}$ such that given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_i |F(t_i) - F(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})| < \varepsilon.$$

We set $(H) \int_a^b f(t)\,dt = F(b) - F(a)$ in this case.

The Henstock integral of an n-dimensional space valued function is defined as follows.

Definition 2.4. A function $f : [a,b] \to \mathbb{R}^n$ is *variationally Henstock integrable* or simply *Henstock integrable* (we write $f \in H([a,b], \mathbb{R}^n)$), if there is a function $F : [a,b] \to \mathbb{R}^n$ such that given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine $d = (\xi_i, t_i) \in TD_{[a,b]}$,

$$\sum_i \|F(t_i) - F(t_{i-1}) - f(\xi_i)(t_i - t_{i-1})\| < \varepsilon.$$

As for the Kurzweil integral, a function $f : [a, b] \to \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$, is *Henstock integrable*, if and only if, every component f_i, $i = 1, \ldots, n$, of f is *Henstock integrable* in the sense of Definition 2.3.

The equality $H([a, b], \mathbb{R}^n) = K([a, b], \mathbb{R}^n)$ holds and we have

$$F(t) - F(a) = (K) \int_a^t f(s)\, ds = \tilde{f}(t) - \tilde{f}(a), \qquad t \in [a, b],$$

(see [8], for instance).

It is easy to see that the above definitions can also be given for functions taking values in a general linear topological space X. However, in a general infinite dimensional space X, $H([a, b], \mathbb{R}^n)$ may be properly contained in $K([a, b], \mathbb{R}^n)$.

In the sequel, we use integration specified by Definitions 2.1 and 2.3: we say that a function in $H([a, b], \mathbb{R}^n) = K([a, b], \mathbb{R}^n)$ is Kurzweil-Henstock integrable with integral $\int_a^b f(t)\, dt$. We simplify the notation by using $\int f(t)dt$ instead of $(K) \int f(t)dt$.

As it should be expected, the Kurzweil-Henstock integral is linear and additive over non-overlapping intervals. If the integral $\int_a^b f(t)\, dt$ exists, then we set $\int_b^a f(t)\, dt = -\int_a^b f(t)\, dt$. Also, $\int_a^b f(t)\, dt = 0$, when $b = a$.

3 Averaging for RFDEs

We start this section by recalling the concept of a regulated function.

Let X be a Banach space and $I \subset \mathbb{R}$ be an interval.

We denote by $G(I, X)$ be the space of locally regulated functions $f : I \to X$, that is, for each compact interval $[a, b] \subset I$, the one-sided limits

$$f(t+) = \lim_{\rho \to 0+} f(t + \rho), \quad t \in [a, b),$$

and

$$f(t-) = \lim_{\rho \to 0-} f(t + \rho), \quad t \in (a, b],$$

exist and are finite. When $I = [a, b]$ we write $G([a, b], X)$ which is a Banach space when endowed with the usual supremum norm, $\| \ \|_\infty$ (see [4]).

In $G(I, X)$ we consider the topology of locally uniform convergence.

By $G^-(I, X)$, we mean the subspace of $G(I, X)$ of left continuous functions for which $f(t-) = \lim_{\rho \to 0-} f(t + \rho) = f(t)$, $t \in I$, except for the left endpoint of I.

Let $PC_1 \subset G^-([-r, \infty), \mathbb{R}^n)$ be an open set with the following property: if y is an element of PC_1 and $\bar{t} \in [-r, \infty)$, then $\bar{y}$ given by

$$\bar{y}(t) = \begin{cases} y(t), & -r \le t \le \bar{t}, \\ y(\bar{t}), & \bar{t} < t \le \infty, \end{cases} \tag{3.1}$$

is also an element of PC_1. In particular, any open ball in $G^-([-r, \infty), \mathbb{R}^n)$ has this property.

Consider the following initial value problem

$$\begin{cases} \dot{y} = f\left(y_t, \dfrac{t}{\varepsilon}\right) \\ y_0 = \phi, \end{cases} \tag{3.2}$$

where $\phi \in G^-([-r,0],\mathbb{R}^n)$ and $\varepsilon > 0$ is a small parameter. We assume that f maps any pair $(\psi, t) \in G^-([-r,0],\mathbb{R}^n) \times [0,\infty)$ to $\mathbb{R}^n$ and that the mapping $t \mapsto f(y_t,t)$ is Kurzweil-Henstock integrable, for all $t \in [0,\infty)$.

Suppose for each $\psi \in G^-([-r,0],\mathbb{R}^n)$, the limit below exists

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\psi, s)\, ds = f_0(\psi), \tag{3.3}$$

where the integral has to be understood in the sense of Kurzweil-Henstock, and consider the averaged RFDE

$$\begin{cases} \dot{y} = f_0(y_t) \\ y_0 = \phi, \end{cases} \tag{3.4}$$

where f_0 is given by (3.3).

If condition (A) is satisfied, then, for every $y \in PC_1$, we have

$$\begin{aligned} \|f_0(y_s) - f_0(y_t)\| &\leq \lim_{T\to\infty}\frac{1}{T}\left\|\int_0^T [f(y_s,\sigma) - f(y_t,\sigma)]d\sigma\right\| \leq \\ &\leq \lim_{T\to\infty}\frac{1}{T}\int_0^T C\|y_s - y_t\|_\infty d\sigma = C\|y_s - y_t\|_\infty \end{aligned} \tag{3.5}$$

which implies that, for $\theta \in [-r,0]$, if y is a solution of (3.4) and $s,t \in [0,\infty)$, with $s \leq t$, then

$$\begin{aligned} \|y(s+\theta) - y(t+\theta)\| &= \left\|\int_{t+\theta}^{s+\theta} f_0(y_\sigma)d\sigma\right\| \leq \\ &\leq \int_{t+\theta}^{s+\theta}\|f_0(y_\sigma) - f_0(0)\|d\sigma + \int_{t+\theta}^{s+\theta}\|f_0(0)\|d\sigma \leq \\ &\leq C\int_{t+\theta}^{s+\theta}\|y_\sigma\|_\infty d\sigma + (t-s)\|f_0(0)\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|y_s - y_t\|_\infty &= \sup_{\theta\in[-r,0]}\|y(s+\theta) - y(t+\theta)\| \leq \\ &\leq C(t-s)\sup_{\sigma\in[s-r,t]}\|y_\sigma\|_\infty + (t-s)\|f_0(0)\|. \end{aligned} \tag{3.6}$$

Thus, as $t - s \to 0$, we have $\|y_s - y_t\|_\infty \to 0$, that is, when y_t is a function of t, for $t \in [0,\infty)$, y_t is a continuous function.

Now, let us present some lemmas which will help us establish our averaging result for RFDEs.

Lemma 3.1. *Consider* (3.3)*. Then, for every $t > 0$ and every $\alpha > 0$, we have*

$$\lim_{\varepsilon\to 0+} \frac{1}{\alpha/\varepsilon} \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds = f_0(\psi), \quad \psi \in G^-([-r, 0], \mathbb{R}^n).$$

Proof. In this proof, we use some ideas borrowed from [6], Lemma 4.2.

By definition,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\psi, s)ds = f_0(\psi), \quad \psi \in G^-([-r, 0], \mathbb{R}^n). \tag{3.7}$$

Thus, clearly, for $t > 0$ and $\alpha > 0$, we also have

$$\lim_{\varepsilon\to 0^+} \frac{1}{t/\varepsilon + \alpha/\varepsilon} \int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds = f_0(\psi) \qquad \text{and}$$

$$\lim_{\varepsilon\to 0^+} \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f(\psi, s)ds = f_0(\psi).$$

Then

$$\lim_{\varepsilon\to 0^+} \left[\frac{1}{t/\varepsilon + \alpha/\varepsilon} \int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds - \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f(\psi, s)ds \right] =$$

$$= \lim_{\varepsilon\to 0^+} \left[\frac{1}{t/\varepsilon + \alpha/\varepsilon} \int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds - f_0(\psi) \right] + \lim_{\varepsilon\to 0^+} \left[f_0(\psi) - \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f(\psi, s)ds \right] = 0.$$

On the other hand,

$$\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds = \frac{1}{\alpha/\varepsilon} \int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds - \frac{1}{\alpha/\varepsilon} \int_0^{t/\varepsilon} f(\psi, s)ds =$$

$$= \left(\frac{t/\varepsilon + \alpha/\varepsilon}{t/\varepsilon + \alpha/\varepsilon} \right) \frac{1}{\alpha/\varepsilon} \int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds - \left(\frac{t/\varepsilon}{t/\varepsilon} \right) \frac{1}{\alpha/\varepsilon} \int_0^{t/\varepsilon} f(\psi, s)ds =$$

$$= \frac{1}{t/\varepsilon + \alpha/\varepsilon} \left(\frac{t}{\alpha} + 1 \right) \int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds - \frac{t}{\alpha} \cdot \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f(\psi, s)ds =$$

$$= \frac{1}{t/\varepsilon + \alpha/\varepsilon} \int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds +$$

$$+ \frac{t}{\alpha} \left[\frac{1}{t/\varepsilon + \alpha/\varepsilon} \int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds - \frac{1}{t/\varepsilon} \int_0^{t/\varepsilon} f(\psi, s)ds \right].$$

Therefore,

$$\lim_{\varepsilon\to 0^+} \left[\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds - f_0(\psi) \right] = \lim_{\varepsilon\to 0^+} \left[\frac{1}{t/\varepsilon + \alpha/\varepsilon} \int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi, s)ds - f_0(\psi) \right] +$$

$$+\lim_{\varepsilon\to 0^+}\frac{t}{\alpha}\left[\frac{1}{t/\varepsilon+\alpha/\varepsilon}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\psi,s)ds-\frac{1}{t/\varepsilon}\int_0^{t/\varepsilon} f(\psi,s)ds\right]=0.$$

Hence

$$\lim_{\varepsilon\to 0^+}\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\psi,s)ds=f_0(\psi)$$

and the result follows. □

The next corollary follows directly from Lemma 3.1.

Corollary 3.1. *Let $t>0$ and $\alpha>0$. Then, for every $y\in PC_1$, we have*

$$\lim_{\varepsilon\to 0+}\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(y_t,s)ds=f_0(y_t).$$

Lemma 3.2. *Let y be a solution of (3.4), where f satisfies condition (A). Then, for every $t>0$, we have*

$$\lim_{\varepsilon\to 0+}\int_0^t f\left(y_s,\frac{s}{\varepsilon}\right)ds=\int_0^t f_0\left(y_s\right)ds.$$

Proof. We borrow some ideas from [6], Lemma 4.4.

Let $\varepsilon>0$ and $t>0$ be given. For $s\geq 0$ and $\psi\in G^-([-r,\infty),\mathbb{R}^n)$, we define

$$f_1(\psi,s)=f(\psi,s)-f_0(\psi).$$

Let δ be a gauge corresponding to $\varepsilon>0$ in the definition of the Kurzweil-Henstock integral $\int_0^t f_1\left(y_\sigma,\frac{\sigma}{\varepsilon}\right)d\sigma$ and consider a δ-fine partition $(\tau_i,[s_i,s_{i+1}])$, $i=0,1,2,\ldots,m-1$, of the interval $[0,t]$. Then,

$$\left\|\int_0^t f\left(y_s,\frac{s}{\varepsilon}\right)ds-\int_0^t f_0\left(y_s\right)ds\right\|=\left\|\int_0^t f_1\left(y_s,\frac{s}{\varepsilon}\right)ds\right\|\leq$$

$$\leq\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}}\left[f_1\left(y_s,\frac{s}{\varepsilon}\right)-f_1\left(y_{s_i},\frac{s}{\varepsilon}\right)\right]ds\right\|+\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}} f_1\left(y_{s_i},\frac{s}{\varepsilon}\right)ds\right\|$$

$$\leq\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}}\left[f\left(y_s,\frac{s}{\varepsilon}\right)-f\left(y_{s_i},\frac{s}{\varepsilon}\right)\right]ds\right\|+\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}}\left[f_0\left(y_s\right)-f_0\left(y_{s_i}\right)\right]ds\right\|+$$

$$+\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}} f_1\left(y_{s_i},\frac{s}{\varepsilon}\right)ds\right\|. \tag{3.8}$$

We can suppose, without loss of generality, that the gauge δ is such that $\delta(\xi)<\frac{\varepsilon}{2}$, for every $\xi\in[0,t]$. Then, by (3.6), for each $i=0,1,2,\ldots,m-1$ and each $s\in[s_i,s_{i+1}]$, we have

$$\begin{aligned}\|y_s-y_{s_i}\|_\infty &< C(s-s_i)\sup_{\sigma\in[s_i-r,s]}\|y_\sigma\|_\infty+(s-s_i)|f_0(0)|<\\ &< C2\delta(\tau_i)\sup_{\sigma\in[s_i-r,s_{i+1}]}\|y_\sigma\|_\infty+2\delta(\tau_i)\|f_0(0)\|<\\ &< C\varepsilon\sup_{\sigma\in[s_i-r,s_{i+1}]}\|y_\sigma\|_\infty+\varepsilon\|f_0(0)\|.\end{aligned}$$

Therefore,

$$\sup_{s\in[s_i,s_{i+1}]}\|y_s-y_{s_i}\|_\infty \le C\varepsilon \sup_{\sigma\in[s_i-r,s_{i+1}]}\|y_\sigma\|_\infty+\varepsilon\|f_0(0)\|$$

which can be made sufficiently small by the arbitrariness of ε, that is, there exists $\eta>0$ sufficiently small such that

$$\sup_{s\in[s_i,s_{i+1}]}\|y_s-y_{s_i}\|_\infty \le \eta, \quad i=0,1,2,\dots,m-1, \tag{3.9}$$

(say, $\eta = C\varepsilon\sup_{\sigma\in[s_i-r,s_{i+1}]}\|y_\sigma\|_\infty+\varepsilon\|f_0(0)\|$).

Then (3.9) and condition (A) imply

$$\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}}\left[f\left(y_s,\frac{s}{\varepsilon}\right)-f\left(y_{s_i},\frac{s}{\varepsilon}\right)\right]ds\right\|\le$$

$$\le\sum_{i=0}^{m-1}\sup_{\sigma\in[s_i,s_{i+1}]}\|y_\sigma-y_{s_i}\|_\infty\int_{s_i}^{s_{i+1}}C ds\le \eta C\sum_{i=0}^{m-1}(s_{i+1}-s_i)=\eta Ct.$$

Since η can be chosen sufficiently small (by the arbitrariness of ε), the first summand on the righthand side of (3.8) tends to zero as $\varepsilon\to 0$.

Now, using (3.5) and (3.9), for each $i=0,1,2,\dots,m-1$ and each $s\in[s_i,s_{i+1}]$, we have

$$\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}}\left[f_0\left(y_s\right)-f_0\left(y_{s_i}\right)\right]ds\right\|\le\sum_{i=0}^{m-1}\int_{s_i}^{s_{i+1}}\left\|f_0\left(y_s\right)-f_0\left(y_{s_i}\right)\right\|ds\le$$

$$\le C\sum_{i=0}^{m-1}\int_{s_i}^{s_{i+1}}\|y_s-y_{s_i}\|_\infty ds\le C\sum_{i=0}^{m-1}\sup_{\sigma\in[s_i,s_{i+1}]}\|y_\sigma-y_{s_i}\|_\infty(s_{i+1}-s_i)\le$$

$$\le \eta C\sum_{i=0}^{m-1}(s_{i+1}-s_i)=\eta Ct.$$

which tends to zero as $\varepsilon\to 0^+$. Therefore the second summand on the righthand side of (3.8) tends to zero as $\varepsilon\to 0^+$.

Finally, we assert that the sum

$$\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}}f_1\left(y_{s_i},\frac{s}{\varepsilon}\right)ds\right\|$$

can be made arbitrarily small by Corollary 3.1. Then this fact will imply that the third summand on the righthand side of (3.8) tends to zero as $\varepsilon\to 0^+$. Let us prove the assertion. We have

$$\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}}f_1\left(y_{s_i},\frac{s}{\varepsilon}\right)ds\right\|=\sum_{i=0}^{m-1}\left\|\int_{s_i}^{s_{i+1}}\left[f\left(y_{s_i},\frac{s}{\varepsilon}\right)-f_0(y_{s_i})\right]ds\right\|=$$

$$= \sum_{i=0}^{m-1} \left\| \int_{s_i}^{s_i+\alpha_i} \left[f\left(y_{s_i}, \frac{s}{\varepsilon} \right) - f_0(y_{s_i}) \right] ds \right\| = \sum_{i=0}^{m-1} \alpha_i \left\| \frac{\varepsilon}{\alpha_i} \int_{s_i/\varepsilon}^{s_i/\varepsilon+\alpha_i/\varepsilon} f\left(y_{s_i}, s \right) ds - f_0(y_{s_i}) \right\|,$$

where $\alpha_i = s_{i+1} - s_i$, for $i = 0, 1, 2, \ldots, m-1$.

Now, for each $i = 0, 1, 2, \ldots, m-1$, define

$$\beta_i = \frac{\varepsilon}{\alpha_i} \int_{s_i/\varepsilon}^{s_i/\varepsilon+\alpha_i/\varepsilon} f\left(y_{s_i}, s \right) ds - f_0(y_{s_i})$$

and let $\beta = \max\{\beta_i;\, i = 0, 1, 2, \ldots, m-1\}$. Then

$$\left\| \sum_{i=0}^{m-1} \alpha_i \beta_i \right\| \leq \beta \sum_{i=0}^{m-1} \alpha_i = \beta \sum_{i=0}^{m-1} (s_{i+1} - s_i) = \beta t.$$

By Corollary 3.1, β can be taken sufficiently small. Therefore the third summand on the righthand side of (3.8) tends to zero as $\varepsilon \to 0^+$ and the result follows. □

Theorem 3.1. *Consider the RFDE* (3.2)*, where* $\phi \in G^-([-r, 0], \mathbb{R}^n)$ *and* f *satisfies condition (A). Consider the averaged RFDE* (3.4)*, where* f_0 *is given by* (3.3)*. Suppose* $[0, \overline{b})$ *is the maximal interval of existence of* (3.2) *and* $[0, b)$ *is the maximal interval of existence of* (3.4)*. Assume that* x^ε *is a maximal solution of* (3.2) *and* y *is a maximal solution of* (3.4)*. Let* $M > 0$*,* $M < \min(\overline{b}, b)$*. Then, for every* $t \in [0, M]$*,*

$$\lim_{\varepsilon \to 0+} \|x^\varepsilon(t) - y(t)\| = 0.$$

Proof. This proof follows the main steps of [6], Lemma 4.5, and uses results from the Kurzweil-Henstock integration theory.

Let $\varepsilon > 0$ and consider $\delta > 0$ be a gauge corresponding to $\varepsilon > 0$ in the definition of the Kurzweil-Henstock integral and consider a δ-fine partition $(\tau_i, [s_i, s_{i+1}])$, $i = 0, 1, 2, \ldots, m-1$ of the interval $[0, M]$. Also, consider $s_0 = 0$. By hypothesis, for each $t \in [0, M]$, the equalities

$$x^\varepsilon(t) = \phi(0) + \int_0^t f\left((x^\varepsilon)_s, \frac{s}{\varepsilon} \right) ds \qquad \text{and} \qquad y(t) = \phi(0) + \int_0^t f_0\left(y_s \right) ds \tag{3.10}$$

hold. In particular, if $t \in [0, s_1] = [s_0, s_1]$ (remember $s_0 = 0$), then the equalities (3.10) hold.

Then, using condition (A), we obtain

$$\|x^\varepsilon(t) - y(t)\| = \left\| \int_0^t \left[f\left((x^\varepsilon)_s, \frac{s}{\varepsilon} \right) - f_0\left(y_s \right) \right] ds \right\| \leq$$

$$\leq \int_0^t \left\| f\left((x^\varepsilon)_s, \frac{s}{\varepsilon} \right) - f\left(y_s, \frac{s}{\varepsilon} \right) \right\| ds + \left\| \int_0^t \left[f\left(y_s, \frac{s}{\varepsilon} \right) - f_0\left(y_s \right) \right] ds \right\| \leq$$

$$\leq \int_0^t C \|(x^\varepsilon)_s - y_s\|_\infty ds + \left\| \int_0^t \left[f\left(y_s, \frac{s}{\varepsilon} \right) - f_0\left(y_s \right) \right] ds \right\|.$$

Using the fact that $(x^\varepsilon)_0 = \phi = y_0$, we have

$$\|(x^\varepsilon)_s - y_s\|_\infty = \sup_{\theta\in[-r,0]} \|x^\varepsilon(s+\theta) - y(s+\theta)\| = \sup_{\sigma\in[0,s]} \|x^\varepsilon(\sigma) - y(\sigma)\|$$

and, therefore,

$$\|x^\varepsilon(t) - y(t)\| \leq \int_0^t C \sup_{\sigma\in[0,s]} \|x^\varepsilon(\sigma) - y(\sigma)\| ds + \left\| \int_0^t \left[f\left(y_s, \frac{s}{\varepsilon}\right) - f_0\left(y_s\right)\right] ds \right\|. \quad (3.11)$$

Since the righthand side of (3.11) is increasing, we have

$$\sup_{\tau\in[0,t]} \|x^\varepsilon(\tau) - y(\tau)\| \leq$$

$$\leq \int_0^t C \sup_{\sigma\in[0,s]} \|x^\varepsilon(\sigma) - y(\sigma)\| ds + \sup_{\tau\in[0,t]} \left\| \int_0^\tau \left[f\left(y_s, \frac{s}{\varepsilon}\right) - f_0\left(y_s\right)\right] ds \right\|.$$

Then, by the Gronwall's inequality for the Kurzweil-Henstock integral (see [11], Corollary 1.43), we obtain

$$\sup_{\tau\in[0,t]} \|x^\varepsilon(\tau) - y(\tau)\| \leq e^{Ct} \sup_{\tau\in[0,t]} \left\| \int_0^\tau \left[f\left(y_s, \frac{s}{\varepsilon}\right) - f_0\left(y_s\right)\right] ds \right\|.$$

Finally,

$$\sup_{\tau\in[0,t]} \left\| \int_0^\tau \left[f\left(y_s, \frac{s}{\varepsilon}\right) - f_0\left(y_s\right)\right] ds \right\|$$

can be taken sufficiently small by Lemma 3.2 and by the compactness of the interval $[0, t]$. The theorem is therefore proved. □

Now, we consider the RFDE

$$\begin{cases} \dot{y} = f\left(y_t, t\right) \\ y_0 = \phi, \end{cases} \quad (3.12)$$

where $\phi \in G^-([-r, 0], \mathbb{R}^n)$, f maps any pair $(\psi, t) \in G^-([-r, 0], \mathbb{R}^n) \times [0, \infty)$ to $\mathbb{R}^n$, the mapping $t \mapsto f(y_t, t)$ is Kurzweil-Henstock integrable, $t \in [0, \infty)$, and f satisfies condition (A).

Let $\varepsilon > 0$ be a small parameter and consider the RFDE

$$\begin{cases} \dot{y} = \varepsilon f\left(y_t, t\right) \\ y_0 = \phi \end{cases} \quad (3.13)$$

Let y be a solution of (3.13). Applying the substitution $\varphi(s) = \frac{s}{\varepsilon}$ (see [11], Theorem 1.18), we have

$$\int_0^{t/\varepsilon} \varepsilon f\left(y_s, s\right) ds = \int_0^t f\left((y)_{\frac{s}{\varepsilon}}, \frac{s}{\varepsilon}\right) ds = \int_0^t f\left(\zeta_s, \frac{s}{\varepsilon}\right) ds, \quad (3.14)$$

where $(\zeta)_{\varepsilon t} = (y)_t$, $t \in [0, \frac{L}{\varepsilon}]$, that is, ζ is a solution on $[0, L]$ of the system

$$\begin{cases} \dot{y} = f\left(y_t, \dfrac{t}{\varepsilon}\right) \\ y_0 = \phi. \end{cases} \tag{3.15}$$

On the other hand, consider $f_0 : G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ given by

$$f_0(\psi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\psi, s)\, ds \tag{3.16}$$

and the averaged autonomous RFDE

$$\begin{cases} \dot{y} = f_0(y_t) \\ y_0 = \phi. \end{cases} \tag{3.17}$$

If $\overline{y}$ is a solution of (3.17) on $[0, L]$, then by Lemma 3.2, we have

$$\lim_{\varepsilon\to 0} \int_0^t f\left(\overline{y}_s, \frac{s}{\varepsilon}\right) ds = \int_0^t f_0(\overline{y}_s) ds, \qquad t \in [0, L]. \tag{3.18}$$

Thus, using (3.14) and (3.18), we obtain the following approximation

$$\left\| \varepsilon \int_0^{\frac{t}{\varepsilon}} f(y_s, s)\, ds - \int_0^t f_0(\overline{y}_s) ds \right\| \approx \left\| \int_0^t f\left(\zeta_s, \frac{s}{\varepsilon}\right) ds - \int_0^t f\left(\overline{y}_s, \frac{s}{\varepsilon}\right) ds \right\|,$$

whenever $\varepsilon > 0$ is sufficiently small.

By condition (A), we have

$$\left\| \int_0^t f\left(\zeta_s, \frac{s}{\varepsilon}\right) ds - \int_0^t f\left(\overline{y}_s, \frac{s}{\varepsilon}\right) ds \right\| \leq C \int_0^t \|\zeta_s - \overline{y}_s\|_\infty ds. \tag{3.19}$$

Then by Theorem 3.1, we conclude that the righthand side of (3.19) can be chosen sufficiently small. Therefore we proved the next corollary.

Corollary 3.2. *Consider the RFDEs* (3.13) *and* (3.17)*. Then for every $\rho > 0$ and every $L > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\left\| \varepsilon \int_0^{\frac{t}{\varepsilon}} f(y_s, s)\, ds - \int_0^t f_0(\overline{y}_s) ds \right\| < \rho, \quad t \in [0, L],$$

where y is a solution of (3.13) *on $[0, \frac{L}{\varepsilon}]$ and $\overline{y}$ is a solution of* (3.17) *on $[0, L]$.*

The next corollary is an immediate consequence of Corollary 3.2.

Corollary 3.3. *Consider the RFDEs* (3.13) *and* (3.17)*. Then for every $\rho > 0$ and every $L > 0$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\|y - \overline{y}\|_\infty < \rho,$$

on $[0, \frac{L}{\varepsilon}]$, where y is a solution of (3.13) *on $[0, \frac{L}{\varepsilon}]$ and $\overline{y}$ is a solution of* (3.17) *on $[0, L]$.*

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